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# Quantization by Parts, Maximal Symmetric Operators, and Quantum Circuits

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In the context of a generalized quantum theory which admits maximal symmetric operators as observables, we discuss a quantization scheme which can systematically deal with what may be called *quantum circuits*. The scheme, known as the method of *quantization by parts*, has recently been applied to obtain a new derivation of the Josephson equation for the supercurrent through a Josephson junction in a superconducting circuit. This paper presents an application of this scheme to several circuit configurations, namely, from one-branch to many-branch circuits. We also propose an experimental test on whether the condensate is always in a pure state, using a three-branch Y-shape circuit.

## 1. INTRODUCTION

In this paper we present a condensate wave function or quasiparticle approach to superconducting circuits championed by Feynman (1965, 1972). We know from BCS theory that the electrons in the ground state forming the condensate in a superconductor come in pairs (Cooper pairs) with opposite spin. According to Feynman, each pair of electrons in the condensate can be treated as a single particle (quasiparticle) of charge $q$ and mass $m$ twice that of an electron; the condensate can be regarded as consisting of a large number of these quasiparticles all in the *same* quantum state (Feynman, 1972). The quantum state of a quasiparticle is assumed to be describable by a one-particle wave function $\psi(x)$ which is normalized to one. We can set up quantum mechanical observables as operators and the Schrödinger equation with an appropriate Hamiltonian in the usual fashion. The simplicity of this approach lies in the fact that we can effectively use this one-particle wave function to represent the whole condensate by normalizing the wave function

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to the total number of quasiparticles. Then $\psi(x)$ represents the entire condensate with $|\psi(x)|^2$ interpreted as the quasiparticle number density in the condensate. This condensate wave function is often referred to as a *macroscopic* wave function (Tilley and Tilley, 1990). As it turns out, the successful application of this approach does not depend on how $\psi(x)$ is normalized, i.e., it is just a matter of convenience how one normalizes $\psi(x)$.

This paper presents a study of various exactly soluble model superconducting circuits with a direct (as opposed to an alternating) supercurrent flowing in them. These models deal with simple but nontrivial systems and are necessarily idealistic. For example, we would ignore any capacitive effect in a Josephson junction. The circuits with alternating supercurrents will be studied in a separate paper. The simplest circuit consists of a *branch* which is assumed to be one-dimensional and hence can be idealized mathematically as a line which could be finite or infinite in length. Generally a quantum circuit contains *branch points*. A branch point is where a branch encounters an abrupt change in its geometry. This could be a point at which a branch splits into several branches or it could be a branch coming into contact with a plane or other circuit elements. One of the simplest quantum circuits is a branch joining up with another branch, but with a thin gap at the joint. Within classical circuit theory no direct current can flow across such a gap without involving a voltage across the gap and an electrical breakdown. A quantum current in the form of a supercurrent is able to flow across the gap by quantum tunneling without needing a voltage across the gap, a phenomenon now known as the Josephson effect (Josephson, 1962), showing the difference between the behavior of quantum circuits and classical circuits.

Recently we introduced a method of *quantization by parts* to deal with superconducting circuits (Wan and Fountain, 1996). The idea is to quantize each branch separately first, and then bring the separately quantized branches together to arrive at a theory for the whole circuit. Our work is built on previous studies into quantum wires, notably by Exner *et al.* (Exner and Seba, 1987, 1989a, 1989b; Exner *et al.*, 1989; Blank *et al.*, 1994). A circuit geometry generally does not have a manifold structure, which renders conventional methods of quantization inapplicable. First, there is a need to extend the set of operators describing observables beyond the orthodox set of self-adjoint operators. We have argued (Wan *et al.*, 1995) that an observable should generally be represented by a maximal symmetric operator which need not necessarily be self-adjoint. This extension of the set of observables will be seen to be particularly relevant to quantum circuits. Second, one often encounters the nonuniqueness problem in quantizing a circuit.

So, our starting point would be a generalized quantum theory which includes maximal symmetric operators as observables (Wan *et al.*, 1995). Our method of quantization by parts consists of three clearly defined stages:

1. *Partial quantization stage*. This involves an initial quantization of separate branches of the circuit. To quantize in a branch, we shall first quantize a classical observable into an operator $\hat{A}_0$ on the set of infinitely differentiable functions of compact support within an appropriate Hilbert space of square-integrable functions. Nonuniqueness arises because $\hat{A}_0$ may have many maximal symmetric extensions. So there may not be a unique maximal symmetric operator to correspond to a given classical observable on each branch of the circuit.

2. *Composite quantization stage*. This involves combining the partially quantized parts into a single quantized system. This is achieved by taking the direct sum of the partially quantized quantities and deriving its self-adjoint or maximal symmetric extensions. This can become a rather intricate and lengthy process. As will be seen later, our present method of quantization also applies to one-branch circuits. For such circuits this composite quantization stage is absent. Again there is the possibility of nonuniqueness emerging. The nonuniqueness that arises from partial and composite quantization turns out to be a blessing in disguise which motivates the injection of fundamental physical considerations to resolve the quantization problem.

3. *Correlations stage*. This is the final stage. Here we have to examine the physical system carefully. The object here is to establish the correct relationship between cognate observables. Additional physical assumptions may be required to complete this final stage successfully. There are general conditions applicable to all systems. In addition, we have to bring the physical properties of the system into the picture in order to establish correlation conditions appropriate to the system. As will be seen presently, even a one-branch circuit requires careful correlations.

In carrying out the correlations stage we should note that classical relations generally do not carry over as operator relations in a straightforward manner. An example is the relationship between the momentum $p$ and the Hamiltonian $H$ of a particle in an infinite square potential well discussed in most textbooks. Classically we have $H = p^2$ inside the well, taking the mass to be one half for simplicity. The Hamiltonian is quantized into a unique self-adjoint operator $\hat{H}$ on a domain consisting of twice-differentiable functions vanishing at the well boundary. On the other hand, the momentum, quantized as the differential expression $-i\hbar\, d/dx$ on $C_0^\infty$, has an infinite number of self-adjoint extensions $\hat{p}_\lambda$ parametrized by a real number $\lambda$ (Akhiezer and Glazman, 1963). Clearly $\hat{H} \neq \hat{p}_\lambda^2$ for any $\lambda$.

In this paper we shall confine our attention to the case where a (constant) direct supercurrent flows in our superconducting systems without a voltage. Then the total current in the circuit is equal to the supercurrent flowing, since there is no normal current component here (Rose-Innes and Rhoderick, 1969).

Such a direct supercurrent is therefore a measurable physical quantity which should be treated as an observable. In other words, we should have a supercurrent observable in addition to the more familiar observables like momentum and energy. For a direct current in a superconducting circuit we shall make two physical assumptions which will be seen to lead to the necessary correlation conditions:

(PA1) A superconducting state with an established supercurrent $I$ corresponds to a (generalized) eigenfunction of the supercurrent operator $\hat{J}$ with the current $I$ equal to the corresponding eigenvalue $j$.

(PA2) A superconducting state with an established supercurrent $I$ must also correspond to a (generalized) eigenfunction of the Hamiltonian of the system.

Assumption (PA1) is in keeping with orthodox quantum mechanical situation that a state corresponding to a definite value $a$ of an observable $\hat{A}$ should be described by the eigenfunction $\varphi_a$ of $\hat{A}$ associated with the eigenvalue $a$. If such a state were to remain the eigenfunction of $\hat{A}$ associated with eigenvalue $a$ as time goes on, then $\varphi_a$ must be an eigenfunction of the Hamiltonian as well. This is the reason for assumption (PA2), which ensures the stability of the supercurrent in a superconducting state.

As we shall see later, (PA1) and (PA2) together lead to superselection rules which distinguish our present superconducting system from a usual one-particle system in orthodox quantum mechanics. Intuitively we can also see that the phenomenon of tunneling manifests itself quite differently for a beam of electrons and for a condensate. Electrons can be arranged to flow though a circuit one by one in metallic nanostructures and they can pass though an insulating barrier by tunneling (Devoret *et al.*, 1992). However, for single-electron tunneling a reflection invariably occurs if the eigenfunction corresponds to an energy eigenvalue less than the barrier height (Exner and Seba, 1987; Mandl, 1992). This is because an eigenfunction of the Hamiltonian is not an eigenfunction of the momentum operator. Consequently such an electron current will experience resistance (Devoret *et al.*, 1992). (PA1) is equivalent to an assumption of no reflection at the barrier for a superconducting state, this being a characteristic feature distinguishing a superconducting state for a condensate from a usual one-particle state for an electron. This assumption is also crucial in determining the kinetic energy operators and in the derivation of Josephson's equation, as will be seen later.

## 2. TWO-BRANCH CIRCUITS

Before we consider one-branch circuits we shall start with the standard, and hence more familiar, two-branch circuit (Feynman, 1965). The method of quantization by parts has recently been successfully applied to to such a system (Wan and Fountain, 1996). We shall briefly review the results here as an introduction to the methodology involved.

Consider two long superconductors linked up through a thin, typically of the order of a nanometer, insulating layer known as a Josephson junction. We shall consider the symmetrical case where the superconductors on either side of the junction are identical and where the junction itself is symmetrical about its center. It is known that at low temperatures when the metal is in a superconducting state a direct current $j$ can tunnel through the junction as a supercurrent without generating a voltage across the insulator junction. It is also known that $j$ has a sinusoidal dependence on the phase difference $\lambda$ in the wave function across the barrier, as given by the Josephson equation $j = j_0 \sin \lambda$, where $j_0$, which is a constant dependent on the physical geometry of the junction, but not on $\lambda$, is the maximum (critical) dc current possible across the junction while still maintaining superconductivity. Let us idealize the geometry of the system as the real line $\mathbb{R} = (-\infty, \infty)$ broken into two half-lines $\mathbb{R}_0^- = (-\infty, 0)$ and $\mathbb{R}_0^+ = (0, \infty)$. We can now proceed with the method of quantization by parts.

## 2.1. Momentum: Partial and Composite Quantization

### *2.1.1. Partial Quantization*

Let us start with the the right-hand branch $B_2$, i.e., $B_2 = \mathbb{R}_0^+$. The Hilbert space associated with this branch is $\mathcal{H}_2 = L^2(B_2) = L^2(\mathbb{R}_0^+)$. It is well known that there does not exist a self-adjoint momentum operator in the Hilbert space $L^2(\mathbb{R}_0^+)$. In other words, starting with the differential operator $-i\hbar d/dx$ acting on $C_0^\infty(\mathbb{R}_0^+)$, we can obtain no self-adjoint extension in $L^2(\mathbb{R}_0^+)$. However, there is a unique maximal symmetric extension which we shall denote by $\hat{p}_+$ (Wan *et al.*, 1995; Akhiezer and Glazman, 1963; Weidman, 1980).

Similarly, for the momentum on the left-hand branch $B_1 = \mathbb{R}_0^-$, for which the Hilbert space is $\mathcal{H}_1 = L^2(B_1) = L^2(\mathbb{R}_0^-)$, we have the maximal symmetric operator $\hat{p}_-$.

To conclude, we shall take the partially quantized momentum for $B_1$ and $B_2$ to be $\hat{p}_1 = \hat{p}_-$ in $\mathcal{H}_1$ and $\hat{p}_2 = \hat{p}_+$ in $\mathcal{H}_2$, respectively, and $\hat{J}_1 = (q/m)\hat{p}_1, \hat{J}_2 = (q/m)\hat{p}_2$ as the partially quantized supercurrent operators (Wan *et al.*, 1995).

The appearance here of symmetric operators which are not self-adjoint may cause some unease in some quarters. We would argue that quantum mechanics should be extended beyond the set of self-adjoint operators for the description of observables. In fact there is now a well-established generalization of quantum mechanics beyond the set of self-adjoint operators (Busch *et al.*, 1995; Schroeck, 1996). The orginal reason for employing self-adjoint operators is that a self-adjoint operator generates a unique projector-valued (PV) measure through the spectral theorem and that this spectral measure provides the basis for the probabilistic interpretation of quantum mechanics. It has been realized for some time that the probabilistic interpretation of quantum mechanics does not necessarily require the restriction to self-adjoint operators. All that is required is positive operator-valued (POV) measures. A PV measure is just a special case of a POV measure. It is a well-known mathematical result that a maximal symmetric operator generates a unique POV measure through a generalized spectral theorem. There is then no reason not to include maximal symmetric operators for the description of observables. Such an extension would legitimize the use of the quantized radial momentum operator, which is maximal symmetric but not self-adjoint. More details can be found in Wan *et al.* (1995), Busch *et al.* (1995), and Schroeck (1996).

### 2.1.2. Composite Quantization

1. The composite momentum is a self-adjoint extension of the direct sum $\hat{P}^{(2)} = \hat{p}_1 \oplus \hat{p}_2$ in $\mathscr{H}^{(2)} = \mathscr{H}_1 \oplus \mathscr{H}_2$. This direct sum has a family of self-adjoint extensions parametrized by a real variable $\lambda$ in the range $(-\pi, \pi]$ and for each $\lambda \in (-\pi, \pi]$ the self-adjoint extension $\hat{P}^{(2)}_\lambda$ is given on the domain (Wan and Fountain, 1996)

$$\mathscr{D}(\hat{P}^{(2)}_\lambda) = \{\phi = \phi_1 \oplus \phi_2 \in \mathscr{H}^{(2)}\colon \phi_s \in AC(B_s),\ d\phi_s/dx \in \mathscr{H}_s,\ s = 1, 2,$$
$$\phi_1(0) = e^{-i\lambda}\phi_2(0)\} \tag{1}$$

by

$$\hat{P}^{(2)}_\lambda = -i\hbar\left(\frac{d\phi_1}{dx} \oplus \frac{d\phi_2}{dx}\right) \qquad \forall \phi \in \mathscr{D}(\hat{P}^{(2)}_\lambda) \tag{2}$$

2. The composite supercurrent is then $\hat{J}^{(2)}_\lambda = (q/m)\,\hat{P}^{(2)}_\lambda$.
3. Generalized eigenfunctions of $\hat{P}^{(2)}_\lambda$ and $\hat{J}^{(2)}_\lambda$ are of the form

$$\varphi^{(2)}_{\lambda p} = \varphi_{1\lambda p} \oplus \varphi_{2\lambda p}, \qquad p \in \mathbb{R} \tag{3}$$

$$\varphi_{1\lambda p} = e^{ipx}, \qquad x \in \mathbb{R}^-_0 \quad \text{and} \quad i = i/\hbar \tag{4}$$

$$\varphi_{2\lambda p} = e^{i\lambda}e^{ipx}, \qquad x \in \mathbb{R}^+_0 \tag{5}$$

with generalized eigenvalues $p^{(2)}_\lambda = p$ and $j^{(2)}_{\lambda p} = qp/m$, respectively. When $\lambda = 0$ these functions are continuous across the junction and the corresponding self-adjoint extension is simply the standard momentum operator $\hat{p} = -i\hbar d/dx$ in $L^2(\mathbb{R})$. Note that $\varphi_{2\lambda p}$ is not a generalized eigenfunction of $\hat{p}_2$ since $\varphi_{2\lambda p}$ does not satisfy the boundary condition at $x = 0$ for functions in the domain of $\hat{p}_2$. Similarly $\varphi_{1\lambda p}$ is not a generalized eigenfunction of $\hat{p}_1$.

## 2.2. Kinetic Energy: Partial and Composite Quantization

For the partially quantized kinetic energy operators we take

$$\hat{K}_{0s} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \qquad \text{on the domain} \quad C_0^\infty(B_s), \quad s = 1, 2 \tag{6}$$

To obtain a kinetic energy operator for the composite system we first take the direct sum $\hat{K}_0^{(2)} = \hat{K}_{01} \oplus \hat{K}_{02}$ and then proceed to extend this operator. $\hat{K}_0^{(2)}$ has deficiency indices (2, 2), and hence has a four-parameter family of self-adjoint extensions (Richtmyer, 1978). Our task is to single out one particular extension. First we desire self-adjoint extensions which are symmetrical with respect to the junction at the origin $x = 0$. This can be shown to narrow the extensions down to a two-parameter family. Moreover, this two-parameter family can be determined by imposing boundary conditions at the junction $x = 0$, specified by two real parameters $a$ and $b$, on the wave functions in the domains of the extension operators as follows:

$$\phi'_{10} = a\phi_{10} + b\phi_{20}, \qquad a, b \in \mathbb{R} \tag{7}$$

$$\phi'_{20} = -a\phi_{20} - b\phi_{10} \tag{8}$$

where the prime denotes spatial differentiation and the subscript 0 denotes values at $x = 0$. We shall refer to the above boundary conditions for the two-branch circuit as (BC2) and denote by $\hat{K}_{a,b}^{(2)}$ the self-adjoint extension determined by each pair $(a, b)$ of parameter values. We should point out that as in the case of an infinite square potential well the kinetic energy and momentum operators do not neccesarily follow the simple classical relationship, i.e., generally we have $\hat{K}_{a,b}^{(2)} \neq (\hat{P}_\lambda^{(2)})^2/2m$.

## 2.3. Correlations: The Third Quantization Stage

Our task is to describe the dc Josephson effect with an established dc supercurrent $I$. First we infer from physical assumption (PA1) that the superconducting state $\Psi_I^{(2)}$, being an eigenfunction of the current operator $\hat{J}_\lambda^{(2)}$ corresponding to the eigenvalue $I$, can be taken to be $\varphi_{\lambda p}^{(2)}$ with $I = j_{\lambda p}^{(2)}$ for some $p$. Next let us see how this helps to determine the kinetic energy operator $\hat{K}_{a,b}^{(2)}$. The crucial point here is that $\varphi_{\lambda p}^{(2)}$ must be in the domain of the as yet undetermined Hamiltonian. Since the kinetic energy $\hat{K}_{a,b}^{(2)}$ is an additive part of the Hamiltonian, $\varphi_{\lambda p}^{(2)}$ must also be in the domain of $\hat{K}_{a,b}^{(2)}$. As with any function in the domain of any of the $\hat{K}_{a,b}^{(2)}$, the function $\varphi_{\lambda p}^{(2)}$ must satisfy boundary conditions (BC2). Note that we formally include generalized eigenfunctions in the relevant domains. (BC2) are local conditions, so the fact that $\varphi_{\lambda p}^{(2)}$ are not square-integrable does not affect the reasoning.

Substituting $\varphi_{\lambda p}^{(2)}$ into (BC2), we get

$$\hbar p = a + b e^{i\lambda} \tag{9}$$

$$\hbar p e^{i\lambda} = -a e^{i\lambda} - b \tag{10}$$

Equating real and imaginary parts these equations yields

$$0 = a + b \cos \lambda \tag{11}$$

$$p = \hbar b \sin \lambda \tag{12}$$

The values 0, $\pi$ for $\lambda$ are excluded here for good reasons (Wan and Fountain, 1996).

Before proceeding further, we make the following physical assumption:

(PA3) The parameter $b$ plays the role of a coupling constant and is characteristic of and unique to the junction. For example, $b$ could depend on the geometry of the junction, and in particular $b$ should be independent of $\lambda$.

It follows from (PA3) that the other parameter $a$ becomes dependent on $\lambda$, i.e., $a = -b \cos \lambda$. We obtain the Josephson equation (Wan and Fountain, 1996)

$$j = j_0 \sin \lambda \tag{13}$$

where

$$j = j_{\lambda p}^{(2)} = \frac{q}{m} p, \qquad j_0 = \frac{q}{m} \hbar b > 0 \qquad \text{on assuming} \quad b > 0 \tag{14}$$

We can identify $j$ with the established current $I$ and $j_0$ with the critical current. The critical current is seen to be characteristic of the junction and independent of $\lambda$. Note that constants $a$, $b$ emerge from the self-adjointness conditions; they are not put in by hand, as it were.

For the sake of clarity we shall make explicit the following general quantum mechanical assumption before proceeding further:

(PA4) Each observable of the superconducting system as a whole with an established current should correspond to a unique operator, up to the usual unitary equivalence.

To apply (PA4) to the momentum, we require the current $I$ to determine the value of $\lambda$ so as to single out a momentum operator $\hat{P}_{\lambda}^{(2)}$. Since the value of $\lambda$ is uniquely determined by $j = I$ only over a range of $\pi$, assumption (PA4) would restrict $\lambda$ to a range of $\pi$. Let us choose the range to be $[-\frac{1}{2}\pi, 0) \cup (0, \frac{1}{2}\pi]$. As it turns out, this restriction also helps to determine the value of $a$ and hence the kinetic energy operator $\hat{K}_{a,b}^{(2)}$. Further analysis of this theory leads naturally to the establishment of a superselection rule (Wan and Fountain, 1996). The gist of the argument will also be seen later in the section dealing with one-branch circuits.

Finally we should mention that there are other important two-branch circuits, e.g., the dc SQUID configurations with a superconducting ring broken into two halves by two Josephson junctions. Such a circuit has recently been investigated by Harrison and Wan (1997).

## 3. ONE-BRANCH CIRCUITS: UNIFORMLY THICK SUPERCONDUCTING RINGS WITH A JOSEPHSON JUNCTION

Apart from the infinite line in one dimension, there are two notable cases: (1) a uniformly thick superconducting ring interrupted by a Josephson junction (JJ), and (2) continuous, uniformly thick superconducting rings without a JJ. These two cases have been investigated using a different scheme (Wan and Harrison, 1993) which requires a number of somewhat ad hoc assumptions. Here we shall examine a uniformly thick superconducting ring with a JJ using the method of quantization by parts. It can be seen that the ad hoc assumptions used previously are not needed in our present method.

Due to quantum tunneling processes, a dc supercurrent can persist around the ring, passing through the JJ without an applied voltage. The JJ is assumed to be symmetrical about its center. Here, we shall for simplicity assume the absence of an external magnetic field. A supercurrent could be set up initially with the help of an external magnetic field. Once the current is set up, the external field could then be removed (Feynman, 1965), so that a steady state could be achieved without the presence of an external magnetic field. To model the system, we start with a classical particle of mass $m$ and charge $q$ constrained to move in the open interval $S^1_c = \{\theta \in (-\pi, \pi)\}$ with the JJ located at $\theta = \pi$. The appropriate Hilbert space $L^2(S^1_c)$ of square-integrable functions on $S^1_c$ with respect to the measure $r\, d\theta$, $r$ being the radius of the ring.

### 3.1. Momentum: Partial and Composite Quantization

We start with the differential operator $-(i\hbar/r)\, d/d\theta$ defined on $C^\infty_0(S^1_c)$. This operator has a one-parameter family of self-adjoint extensions $\hat{P}^{(1)}_\lambda$ with domain (Wan and Harrison, 1993; Fano, 1971)

$$D_\lambda = \{\phi : \phi \in AC(S^1_c),\ \phi_- = e^{i\lambda}\phi_+,\ [-(i\hbar/r)\, d/d\theta]\phi \in L^2(S^1_c)\} \tag{15}$$

where $AC(S^1_c)$ is the set of absolutely continuous functions $\phi(\theta)$ on $S^1_c$, $\lambda$ is a real number in the interval $(-\pi, \pi]$, and $\phi_- = \phi(-\pi)$, $\phi_+ = \phi(\pi)$. The momentum is then quantized as the operator $\hat{P}^{(1)}_\lambda = -(i\hbar/r)\, d/d\theta$ on $D_\lambda$ for some as-yet-undefined value of $\lambda$. The operator possesses a discrete spectrum

$$p^{(1)}_{\lambda,n} = (\hbar/r)(n - \lambda/2\pi)$$

with corresponding eigenfunctions

$$\psi^{(1)}_{\lambda,n}(\theta) = \exp[i(n - \lambda/2\pi)\theta] = \exp[\tfrac{i}{\hbar}p_{\lambda,n}r\theta], \qquad n = 0, \pm 1, \pm 2, \ldots$$

As for the two-branch circuit, we shall normalize plane waves to having an absolute value of 1 everywhere so as to produce plane waves representing an average of one particle per unit volume. This enables us to define the same expression of the supercurrent operator in terms of the momentum. In an earlier paper (Wan and Harrison, 1993) a different normalization is used resulting in a different current/momentum relationship.

We may introduce supercurrent and enclosed magnetic flux operators

$$\hat{J}^{(1)}_{\lambda} = \frac{q}{m}\hat{P}^{(1)}_{\lambda} \qquad \text{and} \qquad \hat{\Phi}^{(1)}_{\lambda} = L\hat{J}^{(1)}_{\lambda} = \frac{2\pi r}{q}\hat{P}^{(1)}_{\lambda} \tag{16}$$

whose eigenfunctions are $\psi^{(1)}_{\lambda,n}(\theta)$ with $j^{(1)}_{\lambda,n} = (n - \lambda/2\pi)\, q\hbar/mr$ and $\Phi^{(1)}_{\lambda,n} = (n - \lambda/2\pi)\Phi_0$ as their respective eigenvalues. Here $\Phi_0 = h/q$ is the unit flux quantum and the self-inductance $L$ is chosen to be $2\pi mr/q^2$. The expression for $L$ here is different from that used previously because we have a different normalization constant for the momentum eigenfunctions. Note that for our circuit with a Josephson junction we shall assume that $\lambda \neq 0$, since $\lambda = 0$ is a condition for a continuous ring without a junction. We shall see later that we should also reject the case with $\lambda = \pi$ (Appendix A).

### 3.2. Kinetic Energy: Partial and Composite Quantization

For the kinetic energy we start with the operator

$$\hat{K}^{(1)}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} = -\frac{\hbar^2}{2mr^2}\frac{d^2}{d\theta^2} \qquad \text{on} \quad C^{\infty}_0(S^1_c)$$

This operator is known to have many self-adjoint extensions (Hudson and Pym, 1980). Our task is to sift out the relevant ones from the host of possible extensions. Two obvious self-adjoint extensions present themselves. The first one is the usual Hamiltonian for the infinite square-well potential obtained by requiring the domain to consist of functions vanishing at $\theta = -\pi, \pi$, in addition to differentiability. This extension is not suitable, as it is incompatible with $\hat{P}^{(1)}_{\lambda}$ in the sense that the eigenfunctions of $\psi^{(1)}_{\lambda,n}(\theta)$ of $\hat{P}^{(1)}_{\lambda}$ are not even in the domain of this extension. The second obvious self-adjoint extension is simply $\hat{P}^{(1)}_{\lambda})^2/2m$. This is also not suitable by a symmetry consideration which will be discussed presently.

Let $\mathcal{D}$ be the set of absolutely continuous functions $\phi$ on $\mathcal{S}_c^1$ such that

$$\frac{d\phi}{d\theta} \in AC(\mathcal{S}_c^1), \qquad \frac{d^2\phi}{d\theta^2} \in L^2(\mathcal{S}_c^1)$$

Let $\underline{\alpha} = \{\alpha'_-, \alpha_-, \alpha'_+, \alpha_+\}$ and $\underline{\beta} = \{\beta'_-, \beta_-, \beta'_+, \beta_+\}$ be two sets of complex numbers subject to two conditions:

(C1) The first set is not a multiple of the second set, i.e., there is no number $\delta$ such that $\alpha'_- = \delta\beta'_-$, $\alpha_- = \delta\beta_-$, $\alpha'_+ = \delta\beta'_+$, and $\alpha_+ = \delta\beta_+$.

(C2) These complex numbers are related by

$$\alpha'^*_-\alpha_- - \alpha^*_-\alpha'_- = \alpha'^*_+\alpha_+ - \alpha^*_+\alpha'_+, \qquad \beta'^*_-\beta_- - \beta^*_-\beta'_- = \beta'^*_+\beta_+ - \beta^*_+\beta'_+$$

$$\alpha'^*_-\beta_- - \alpha^*_-\beta'_- = \alpha'^*_+\beta_+ - \alpha^*_+\beta'_+, \qquad \beta'^*_-\alpha_- - \beta^*_-\alpha'_- = \beta'^*_+\alpha_+ - \beta^*_+\alpha'_+$$

Finally, let $\mathcal{D}_{\underline{\alpha},\underline{\beta}}$ be a subset of $\mathcal{D}$ consisting of functions $\phi$ satisfying the following boundary conditions:

$$\alpha'_-\phi'_- - \alpha_-\phi_- = \alpha'_+\phi'_+ - \alpha_+\phi_+ \tag{17a}$$

$$\beta'_-\phi'_- - \beta_-\phi_- = \beta'_+\phi'_+ - \beta_+\phi_+ \tag{17b}$$

where $\phi'_-$ and $\phi'_+$ are the derivatives of $\phi$ with respect to $\theta$ evaluated at $\theta = -\pi$ and $\theta = \pi$, respectively. These boundary conditions for the one-branch circuit are referred to as (BC1). All self-adjoint extensions of $\hat{K}_0^{(1)}$ are given by the following known theorem (Hudson and Pym, 1980):

*Theorem 1.* The operator $\hat{K}^{(1)}_{\underline{\alpha},\underline{\beta}}$ defined on the domain $\mathcal{D}_{\underline{\alpha},\underline{\beta}}$ by

$$\hat{K}^{(1)}_{\underline{\alpha},\underline{\beta}}\phi = -\frac{\hbar^2}{2mr^2}\frac{d^2\phi}{d\theta^2} \qquad \forall\phi \in \mathcal{D}_{\underline{\alpha},\underline{\beta}}$$

is self-adjoint, and conversely every self-adjoint extension of $\hat{K}_0^{(1)}$ is of this form.

First we shall slim down the family of extensions by symmetry considerations. In view of the symmetry of the superconducting ring and the junction assumed at the outset there should be no preferred direction of current flow either clockwise or anticlockwise. In other words, the dynamics and hence the Hamiltonian must be invariant with respect to the direction of the current flow. It follows that the Hamiltonian must have reflection symmetry and therefore should commute with the parity opeator $\hat{\wp}$ defined by $(\hat{\wp}\phi)(\theta) = \phi(-\theta)$. This means that with respect to the parity operator its domain must be invariant. The domain of $(\hat{P}_\lambda^{(1)})^2/2m$ is not invariant. As in the case of the two-branch circuit, this extension is hence not suitable. We would also contend that the above symmetry argument applies to the kinetic energy so that its domain should also be invariant with respect to the parity operator, i.e., we look for $\hat{K}^{(1)}_{\underline{\alpha},\underline{\beta}}$ with appropriate $\underline{\alpha}$, $\underline{\beta}$ so that $\phi \in \mathcal{D}_{\underline{\alpha},\underline{\beta}} \Rightarrow \hat{\wp}\phi \in \mathcal{D}_{\underline{\alpha},\underline{\beta}}$. Hence, in addition to boundary conditions (BC1) we must also have, $\forall \phi \in \mathcal{D}_{\underline{\alpha},\underline{\beta}}$,

$$\begin{Bmatrix} -\alpha'_-\phi'_+ - \alpha_-\phi_+ = -\alpha'_+\phi'_- - \alpha_+\phi_- \\ -\beta'_-\phi'_+ - \beta_-\phi_+ = -\beta'_+\phi'_- - \beta_+\phi_- \end{Bmatrix}$$

$$\text{or} \quad \begin{Bmatrix} -\alpha'_+\phi'_- - \alpha_+\phi_- = -\alpha'_-\phi'_+ - \alpha_-\phi_+ \\ -\beta'_+\phi'_- - \beta_+\phi_- = -\beta'_-\phi'_+ - \beta_-\phi_+ \end{Bmatrix}$$

These equations seemingly correspond to two new sets of parameters $\tilde{\underline{\alpha}} = (-\alpha'_+, \alpha_+, -\alpha'_-, \alpha_-)$ and $\tilde{\underline{\beta}} = (-\beta'_+, \beta_+, -\beta'_-, \beta_-)$. These sets of parameters also satisfy (C1) and (C2). For the same self-adjoint extension these sets must be proportional to the previous sets, i.e., we must have

$$\text{either} \quad \tilde{\underline{\alpha}} = \delta\underline{\alpha}, \quad \tilde{\underline{\beta}} = \delta'\underline{\beta} \qquad \text{or} \qquad \tilde{\underline{\alpha}} = \delta\underline{\beta}, \quad \tilde{\underline{\beta}} = \delta'\underline{\alpha}$$
$$\textit{for some} \quad \delta, \delta' \in \mathbb{C} \tag{18}$$

It turns out that only the second alternative is the correct choice (see Appendix A) and we can, without loss of generality, set the proportionality constants to 1, i.e., we have

$$\alpha'_- = -\beta'_+, \qquad \alpha_- = \beta_+, \qquad \alpha'_+ = -\beta'_-, \qquad \alpha_+ = \beta_- \tag{19}$$

Conditions (C2) reduce to

$$\alpha'^*_-\alpha_- - \alpha^*_-\alpha'_- = \alpha'^*_+\alpha_+ - \alpha^*_+\alpha'_+,$$

$$\alpha'^*_+\alpha_- + \alpha^*_+\alpha'_- = \alpha'^*_-\alpha_+ + \alpha^*_-\alpha'_+$$

This reduces boundary conditions (BC1) to

$$\alpha'_-\phi'_- - \alpha_-\phi_- = \alpha'_+\phi'_+ - \alpha_+\phi_+ \tag{20}$$

$$\alpha'_+\phi'_- + \alpha_+\phi_- = \alpha'_-\phi'_+ + \alpha_-\phi_+ \tag{21}$$

In Appendix A we show that these can be rearranged into the form of (BC2), i.e.,

$$\phi'_- = a\phi_- + b\phi_+, \qquad a, b \in \mathbb{R} \tag{22}$$

$$\phi'_+ = -a\phi_+ - b\phi_- \tag{23}$$

Everything is highly nonunique so far. We must proceed to the correlations stage as we did for the two-branch circuit earlier to try to resolve this nonuniqueness problem.

### 3.3. Correlations Stage

The boundary conditions for the kinetic energy operator are seen to be the same as conditions (BC2) for the two-branch circuit. We can therefore derive the Josephson equation as before by requiring the eigenfunctions of the supercurrent operator $\psi_{\lambda,n}$ to satisfy the above boundary conditions:

$$\psi^{(1)'}_{\lambda,n-} = a\psi^{(1)}_{\lambda,n-} + b\psi^{(1)}_{\lambda,n+}, \qquad \psi^{(1)'}_{\lambda,n+} = -a\psi^{(1)}_{\lambda,n+} - b\psi^{(1)}_{\lambda,n-} \qquad a, b \in \mathbb{R} \tag{24}$$

$$\Rightarrow \bar{\imath} rp_{\lambda,n} = a + be^{i\lambda}, \qquad \bar{\imath} rp_{\lambda,n}\, e^{i\lambda} = -ae^{i\lambda} - b \tag{25}$$

$$\Rightarrow 0 = a + b\cos\lambda, \qquad rp_{\lambda,n} = \hbar b \sin\lambda \tag{26}$$

Assuming the parameter $b$, which is again seen to play the role of a coupling constant, to be characteristic of and unique to the junction, the other parameter $a$ becomes dependent on $\lambda$ and we obtain the Josephson equation

$$j = j_0 \sin\lambda \tag{27}$$

where

$$j = j^{(1)}_{\lambda,n} = \frac{q}{m} p, \qquad j_0 = \frac{q}{mr} \hbar b > 0 \qquad \text{on assuming} \quad b > 0 \tag{28}$$

The Josephson equation can be rewritten as

$$n - \frac{\lambda}{2\pi} = b \sin\lambda \tag{29}$$

We can now identify $j$ above with the established current $I$, and $j_0$ with the critical current. The critical current is seen to be characterisitc of the junction and independent of $\lambda$.

In passing we should point out that there is an existing procedure to derive the Josephson equation for this one-branch circuit through a process of energy minimization (Gough, 1991; Wan and Harrison, 1993).[2] Unlike our present scheme of quantization by parts, which is applicable to different circuit configurations, this process is not generally applicable; the previous two-branch circuit is a case in point.

### 3.4. Constraint and Superselection Rule

For our present superconducting system the state is set up by the supercurrent, which can be established by various physical means. The Josephson equation is a constraint relating the supercurrent to various parameters.

[2] There is a misprint in equation (14) of Gough (1991), which is missing a term to proportional to $-\cos(2\pi\, \Phi_T/\Phi_0)$.

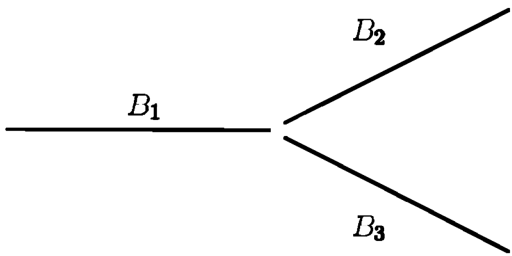
In particular, assumption (PA4) again restricts the phase $\lambda$ to the range $[-\pi/2, 0) \cup (0, \pi/2]$ to establish a one-to-one correspondence between the current and the phase. To sum up we have:

1. The critical current $j_0$ is related to the physical nature of the junction, and $j_0$ determines the parameter $b$.
2. The current $j$ together with $j_0$ determines $\lambda$, and hence determines the parameter $a$. The phase $\lambda$ together with $j_0$ also determines $j$.
3. The two items $j$ and $j_0$ determine operators $\hat{P}^{(1)}_{\lambda}$, $\hat{J}^{(1)}_{\lambda}$, and $\hat{\Phi}^{(1)}_{\lambda}$.
4. The two parameters $a$, $b$ determine the original parameters $\alpha$, $\beta$, and hence the kinetic energy operator $\hat{K}^{(1)}_{\alpha,\beta}$, which may be relabeled $\hat{K}^{(1)}_{a,b}$.
5. The current also determines the state of the system $\psi^{(1)}_{\lambda,n}$.

The consequence of all these constraints is the emergence of a superselection rule. Let us consider the case where an established supercurrent $I$ is equal to the eigenvalue $j^{(1)}_{\lambda_0,0}$, namely with $n = 0$, of the supercurrent operator $\hat{J}^{(1)}_{\lambda_0}$, where $j^{(1)}_{\lambda_0,0}$ and $\lambda_0$ are related by the Josephson equation $j^{(1)}_{\lambda_0,0} = j_0 \sin \lambda_0$. A careful examination of the above constraints reveals that it is inconsistent to take $\hat{J}^{(1)}_{\lambda_0}$ as the supercurrent operator of the system. The operator $\hat{J}^{(1)}_{\lambda_0}$ possesses other eigenvalues $j^{(1)}_{\lambda_0,n}$, $n \neq 0$, corresponding to different values of momentum $p_{\lambda_0,n}$. But these other eigenvalues $j^{(1)}_{\lambda_0,n}$ with $n \neq 0$ are incompatible with the Josephson equation, i.e., $j^{(1)}_{\lambda_0,n} \neq j_0 \sin \lambda_0$. A new current $j^{(1)}_{\lambda_0,n}$ should correspond to a new phase $\lambda_n$ according to the Josephson equation, and consequently a new operator $\hat{J}^{(1)}_{\lambda_n}$ with $j^{(1)}_{\lambda_n,n} = j_0 \sin \lambda_n$, provided $\lambda_n$ exists to satisfy the equation. This process can go on. We would end up with a situation where not all eigenvalues of the current operator $\hat{J}^{(1)}_{\lambda_0}$ are admissible, a result in contradiction with basic orthodox quantum mechanical assumptions. We can resolve this difficulty by recognizing that our system with a given supercurrent is a one-state system. The appropriate Hilbert space is really the generalized subspace spanned by the generalized eigenfunction $\psi^{(1)}_{\lambda_0,0}$, where $\lambda_0$ is related to the current by the Josephson equation. Let us denote this subspace by $\mathcal{H}^{(1)}_I$. The appropriate supercurrent operator is really the restriction $\hat{J}^{(1)}_I$ of $\hat{J}^{(1)}_{\lambda_0}$ to the subspace $\mathcal{H}^{(1)}_I$. The same argument applies to the momentum and the kinetic energy operators. In other words, the appropriate momentum and kinetic energy operators are respectively the restrictions $\hat{P}^{(1)}_I$ of $\hat{P}^{(1)}_{\lambda}$ and $\hat{K}^{(1)}_I$ of $\hat{K}^{(1)}_{\lambda}$ to $\mathcal{H}^{(1)}_I$. There is no coherent superposition of states corresponding to different values of the supercurrent. The situation is similar to that of the two-branch circuit; systems with different currents can be accommodated by taking the appropriate direct integrals. Hence the system possesses a continuous superselection rule. Explicit expressions of these direct integrals are set out in Wan and Harrison (1993).

## 4. THREE-BRANCH CIRCUITS

A typical example is a Y-shape circuit configuration with a Josephson junction preventing direct contact of the three long branches as shown in the following diagram:




Such a circuit has some very interesting features. We aim to present a systematic study here.

### 4.1. Momentum Operators

#### *4.1.1. Partial Quantization*

It is envisaged that a supercurrent flows into the circuit from the far left into branch $B_1$. Branch $B_1$ is therefore chosen to correspond to $\mathbb{R}_0^-$ and the Hilbert space is therefore $\mathcal{H}_1 = L^2(B_1) = L^2(\mathbb{R}_0^-)$. Branches $B_2$, $B_3$ are chosen to correspond to $\mathbb{R}_0^+$ and the Hilbert spaces are therefore chosen to be $\mathcal{H}_2 = L^2(B_2) = L^2(\mathbb{R}_0^+)$, $\mathcal{H}_3 = L^2(B_3) = L^2(\mathbb{R}_0^+)$. The corresponding partially quantized momenta are respectively $\hat{p}_1 = \hat{p}_-$ in $\mathcal{H}_1$, $\hat{p}_2 = \hat{p}_+$ in $\mathcal{H}_2$, and $\hat{p}_3 = \hat{p}_+$ in $\mathcal{H}_3$. These three operators are all maximal symmetric with deficiency indices (0, 1), (1, 0), and (1, 0), respectively.

#### *4.1.2. Composite Quantization*

As the composite momentum we take the direct sum

$$\hat{P}_0^{(3)} = \hat{p}_1 \oplus \hat{p}_2 \oplus \hat{p}_3 \qquad \text{in} \quad \mathcal{H}^{(3)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \tag{30}$$

Since $\hat{P}_0^{(3)}$ has deficiency indices (2, 1) it follows that $\hat{P}_0^{(3)}$ has no self-adjoint extensions (Blank *et al.*, 1994) and that all closed symmetric extensions are merely maximal symmetric. We can work out all these extensions (for details see Appendix B). First we note that all these extensions are restrictions of the adjoint, $\hat{P}_0^{(3)\dagger} = \hat{p}_1^\dagger \oplus \hat{p}_2^\dagger \oplus \hat{p}_3^\dagger$, of $\hat{P}_0^{(3)}$. The domain of the adjoint operator is $\mathfrak{D}(\hat{P}_0^{(3)\dagger}) = \mathfrak{D}(\hat{p}_1^\dagger) \oplus \mathfrak{D}(\hat{p}_2^\dagger) \oplus \mathfrak{D}(\hat{p}_3^\dagger)$. Furthermore, it can be shown (Appendix B) that the extensions are characterizable by restrictions of $\hat{P}_0^{(3)\dagger}$ to domains satisfying certain boundary conditions on the wave functions at the junction.

These boundary conditions are determined by three real parameters $\underline{\lambda} = (\alpha, \lambda_2, \lambda_3)$, as follows (Appendix B):

$$\phi_{20} = \sqrt{1-\alpha^2}\, e^{i\lambda_2}\phi_{10}, \qquad \lambda_2 \in (-\pi, \pi] \tag{31}$$

$$\phi_{30} = \alpha e^{i\lambda_3}\phi_{10}, \qquad \lambda_3 \in (-\pi, \pi] \tag{32}$$

where $\alpha \in [0, 1]$, $\phi_l(x_l) \in \mathcal{H}_l$, $\phi_{l0}, = \phi_l(0)$, and $l = 1, 2, 3$. In other words we have a three-parameter family of maximal symmetric extensions which we shall denote by $\hat{P}^3_{\underline{\lambda}}$. The generalized eigenfuctions of $\hat{P}^{(3)}_{\underline{\lambda}}$ are of the form (Appendices B and C)

$$\varphi^{(3)}_{\underline{\lambda}p} = e^{ipx_1} \oplus \sqrt{1-\alpha^2}\, e^{i\lambda_2} e^{ipx_2} \oplus \alpha e^{i\lambda_3}\, e^{ipx_3}, \qquad x_l \in B_l \tag{33}$$

Since the circuit is assumed symmetrical in $B_2$ and $B_3$; we seek extensions which are invariant with respect to the interchange of $B_2$ and $B_3$. Clearly such extensions correspond to the choice $\alpha = 2^{-1/2}$, $\lambda_2 = \lambda_3 = \lambda \in (-\pi, \pi]$. In other words the boundary conditions are

$$\phi_{20} = \frac{1}{\sqrt{2}} e^{i\lambda}\, \phi_{10} \tag{34}$$

$$\phi_{30} = \frac{1}{\sqrt{2}} e^{i\lambda}\, \phi_{10} \tag{35}$$

The resulting maximal symmetric extensions, to be denoted by $\hat{P}^{(3)}_{\lambda}$, possesses generalized eigenfunctions of the form

$$\varphi^{(3)}_{\lambda p} = e^{ipx_1} \oplus \frac{1}{\sqrt{2}} e^{i\lambda} e^{ipx_2} \oplus \frac{1}{\sqrt{2}} e^{i\lambda} e^{ipx_3} \tag{36}$$

$\hat{P}^{(3)}_{\lambda}$ should not to be confused with the general expression $\hat{P}^{(3)}_{\underline{\lambda}}$. Also, $\varphi^{(3)}_{\lambda p}$ should not to be confused with the general expression $\varphi^{(3)}_{\underline{\lambda}p}$.

For later discussions we shall also require the following two asymmetrical cases:

1. *Case 1* when $\alpha = 0$ with the resulting extension $\hat{P}^{(3)}_{\lambda_2} = \hat{P}^{(2)}_{\lambda_2} \oplus \hat{p}_3$. Here $\hat{P}^{(2)}_{\lambda_2}$ is a self-adjoint extension of $\hat{p}_1 \oplus \hat{p}_2$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ derived earlier for the two-branch circuit. We have

$$\phi_{20} = e^{i\lambda_2}\phi_{10}, \quad \lambda_2 \in (-\pi, \pi] \tag{37}$$

$$\phi_{30} = 0 \tag{38}$$

Generalized eigenfunctions of $\hat{P}^{(3)}_{\lambda_2}$ are of the form

$$\varphi^{(3)}_{\lambda_2 p} = e^{ipx_1} \oplus e^{i\lambda_2}\, e^{ipx_2} \oplus O_3 \quad \text{where } O_3 \text{ is the zero function in } \mathcal{H}_3 \tag{39}$$

Note that $\hat{p}_3$ has no generalized eigenfunction in $\mathcal{H}_3$ because of the condition $\phi_{30} = 0$.

2. *Case 2* when $\alpha = 1$ with the resulting extension denoted by $\hat{P}^{(3)}_{\lambda_3}$. This is the same as case 1 in that $\hat{P}^{(3)}_{\lambda_3}$ is just the direct sum of the self-adjoint extension $\hat{P}^{(2)}_{\lambda_3}$ of $\hat{p}_1 \oplus \hat{p}_3$ in $\mathcal{H}_1 \oplus \mathcal{H}_3$ and $\hat{p}_2$ in $\mathcal{H}_2$. We have

$$\phi_{20} = 0 \tag{40}$$

$$\phi_{30} = e^{i\lambda_3}\phi_{10}, \qquad \lambda_3 \in (-\pi, \pi] \tag{41}$$

Generalized eigenfunctions of $\hat{P}^{(3)}_{\lambda_3}$ are of the form

$$\varphi^{(3)}_{\lambda_3 p} = e^{ipx_1} \oplus O_2 \oplus e^{i\lambda_3} e^{ipx_3}, \qquad \text{where } O_2 \text{ is the zero function in } \mathcal{H}_2 \tag{42}$$

We have to proceed to the correlations stage in order to be in a position to make a final choice of the momentum operator. In the present case the other two related observables are the supercurrent and the kinetic energy.

## 4.2. Supercurrent Operators

### 4.2.1. *Partial and Composite Quantization*

The objective here is to introduce a current operator for the circuit as a whole. The partially quantized supercurrent operators for the branches are $\hat{J}_l = (q/m)\hat{p}_l$. The supercurrent operator for the circuit as a whole is $\hat{J}^{(3)}_{\underline{\lambda}} = (q/m)\, \hat{P}^{(3)}_{\underline{\lambda}}$, which admits $\varphi^{(3)}_{\underline{\lambda}p}$ as an eigenfunction with expected eigenvalue $j^{(3)}_{\underline{\lambda}p} = (q/m)p$. The question arises as to how one would interpret such an eigenfunction. The function $\varphi^{(3)}_{\underline{\lambda}p}$ is a direct sum of three plane waves. What is new here is that these plane waves for the branches do not all normalize to having an absolute value of one. Since the expression $j = (q/m)p$ relating current and momentum is based on a plane wave normalized to an absolute value one, we have to take the normalization factor into account in our interpretation of $\varphi^{(3)}_{\underline{\lambda}p}$. In other words, we should interpret $\varphi^{(3)}_{\underline{\lambda}p}$ as representing a state with a current $j_1 = (q/m)p$ in branch $B_1$, a current $j_2 = (1 - \alpha^2)(q/m)p$ in $B_2$, and $j_3 = \alpha^2\,(q/m)p$ in $B_3$. This interpretation is consistent with current conservation since $B_2$ and $B_3$ are in parallel. We are interested in the following special cases:

1. *The symmetrical case with* $\alpha = 2^{-1/2}$, $\lambda_2 = \lambda_3 = \lambda$. The functions $\varphi^{(3)}_{\lambda p}$ do not vanish on any branch. The momentum operator is $\hat{P}^{(3)}_{\lambda}$ and the supercurrent operator for the circuit becomes $\hat{J}^{(3)}_{\lambda} = (q/m)\, \hat{P}^{(3)}_{\lambda}$ with eigenvalues $j^{(3)}_{\lambda p} = (q/m)p$. This corresponds to an incoming current $I$ that splits equally into branches $B_2$ and $B_3$.

2. *The asymmetrical case with* $\alpha = 0$. The functions $\varphi^{(3)}_{\lambda_2 p}$ vanish on $B_3$. The momentum operator is $\hat{P}^{(3)}_{\lambda_2}$ and the supercurrent operator for the circuit becomes $\hat{J}^{(3)}_{\lambda_2} = (q/m)\hat{P}^{(3)}_{\lambda_2}$ with eigenvalues $j^{(3)}_{\lambda_2 p} = (q/m)p$. This corresponds

to an incoming current $I$ in $B_1$ which goes straight through to $B_2$ with no current going into $B_3$.

3. *The asymmetrical case with* $\alpha = 1$. The functions $\varphi^{(3)}_{\lambda_3 p}$ vanish on $B_2$. The momentum operator is $\hat{P}^{(3)}_{\lambda_3}$ and the supercurrent operator for the circuit becomes $\hat{J}^{(3)}_{\lambda_3} = (q/m)\hat{P}^{(3)}_{\lambda_3}$ with eigenvalues $j^{(3)}_{\lambda_3 p} = (q/m)p$. This corresponds to an incoming current $I$ in $B_1$ which goes solely through to $B_3$ with no current going into $B_2$.

### 4.3. Kinetic Energy Operators

We can adopt as the partially quantized kinetic energy operators in the branches

$$\hat{K}_{0l} = -\frac{\hbar^2}{2m}\frac{d^2}{dx_l^2} \qquad \text{on} \quad C_0^\infty(B_l), \qquad l = 1, 2, 3$$

For composite quantization we construct the direct sum

$$\hat{K}^{(3)}_0 = \hat{K}_{01} \oplus \hat{K}_{02} \oplus \hat{K}_{03} \qquad \text{defined on}$$

$$\mathcal{D}(\hat{K}^{(3)}_0) = \mathcal{D}(\hat{K}_{01}) \oplus \mathcal{D}(\hat{K}_{02}) \oplus \mathcal{D}(\hat{K}_{03}) \tag{43}$$

The deficiency indices of $\hat{K}^{(3)}_0$ are (3, 3), and so $\hat{K}^{(3)}_0$ possesses a nine-parameter family of self-adjoint extensions (Exner and Seba, 1987; Richtmyer, 1978).

It would be tedious as well as pointless to list all the extensions. It is sufficient to consider only those extensions which can be correlated with the momentum and current operators obtained earlier. There are three cases of interest:

1. The symmetrical case with extensions invariant with respect to the interchange of $B_2$ and $B_3$. We can no longer rely on our previous results for two-branch circuits to obtain these self-adjoint extensions. We have to start afresh and go through the lengthy process to find these extensions. Details are presented in Appendix D. The idea is as follows. First the requirement for invariance with respect to the permutation of $B_2$ and $B_3$ reduces the nine-parameter family to a two-parameter family of relevant kinetic energy operators. This two-parameter family can be specified by boundary conditions on the wave functions at the junction, i.e., at $x = 0$ characterized by two arbitrary real parameters $a, b$ on the domains of the extension operators in a way similar to (BC2) for two-branch circuits. We shall refer to these as (BC3):

$$\Phi'_{10} = a\Phi_{10} + b(\Phi_{20} + \Phi_{30}), \qquad a, b \in \mathbb{R} \tag{44}$$

$$\Phi'_{20} + \Phi'_{30} = -a(\Phi_{20} + \Phi_{30}) - 2b\Phi_{10} \tag{45}$$

where a prime represents differentiation with respect to the appropriate position variable. We shall denote the resulting extensions by $\hat{K}^{(3)}_{a,b}$.

2. The asymmetrical case with $\alpha = 0$. The momentum operator is $\hat{P}^{(3)}_{\lambda_2} = \hat{P}^{(2)}_{\lambda_2} \oplus \hat{p}_3$ and the current operator becomes $\hat{J}^{(3)}_{\lambda_2} = (q/m)(\hat{P}^{(2)}_{\lambda_2} \oplus \hat{p}_3)$. The system is acting like a two-branch circuit since there is no current in $B_3$. It follows that we would obtain similar extensions by proceeding as follows:

(a) First we consider a self-adjoint extension $K^{(2)}_{a,b}$ of $\hat{K}_{01} \oplus \hat{K}_{02}$ for $B_1$ and $B_2$.

(b) Then we take the direct sum of $K^{(2)}_{a,b}$ with a self-adjoint extension of $\hat{K}_{03}$ in $\mathcal{H}_3^+$, a natural choice being $\hat{K}_3 = (1/2m)\, \hat{p}_3^\dagger \hat{p}_3$, i.e., we get $\hat{K}^{(3)1,2}_{a,b} = \hat{K}^{(2)}_{a,b} \oplus \hat{K}_3$.

3. The asymmetrical case with $\alpha = 1$. This is the same as above except with branches $B_2$ and $B_3$ interchanged. Choose the kinetic energy operator to be the direct sum of $\hat{K}^{(2)}_{a,b}$ in $\mathcal{H}_1 \oplus \mathcal{H}_3$ and $\hat{K}_2 = (1/2m)\hat{p}_2^\dagger$ in $\mathcal{H}_2$. The resulting extensions are denoted by $\hat{K}^{(3)1,3}_{a,b}$.

## 4.4. Correlations

The physical phenomenon under investigation is the dc Josephson effect. Our task is to describe the system with an established dc supercurrent. We shall proceed as we did for the two-branch circuit. Again we examine three special cases:

1. *The symmetrical case with* $\alpha = 2^{-1/2}$, $\lambda_2 = \lambda_3 = \lambda$. Substituting $\varphi^{(3)}_{\lambda p}$ into the boundary conditions (BC3), we obtain

$$i\!\!\!\bar{}\,p = a + \sqrt{2}be^{i\lambda} \tag{46}$$

$$i\!\!\!\bar{}\,pe^{i\lambda} = -ae^{i\lambda} - \sqrt{2}b, \qquad \text{provided} \quad \lambda \neq 0, \pi \tag{47}$$

When $\lambda = 0, \pi$ the function $\varphi^{(3)}_{\lambda p}$ does not satisfy boundary conditions (BC3) and should be excluded.

Equating real and imaginary parts of the above equations yields

$$0 = a + \sqrt{2}b \cos \lambda \tag{48}$$

$$p = \hbar \sqrt{2}b \sin \lambda \tag{49}$$

As with the two-branch circuit, we now make the following assumption:

(PA3b) The parameter $\sqrt{2}b$, which plays the role of a coupling constant, is characteristic of and unique to the junction, for example, $\sqrt{2}b$ could depend on the geometry of the junction and in particular $\sqrt{2}b$ should be independent of $\lambda$.

It follows that the other parameter $a$ is dependent on $\lambda$, i.e., $a = -\sqrt{2b}\cos\lambda$. Equation (48) then gives the Josephson equation, as it is equivalent to

$$j = j_0 \sin\lambda \tag{50}$$

where

$$j = j^{(3)}_{\lambda p} = \frac{q}{m}p, \qquad j_0 = \frac{q}{m}\hbar\sqrt{2b} > 0 \qquad \text{on assuming} \quad b > 0 \tag{51}$$

We can now identify $j$ above with the established current $I$, and $j_0$ with the critical current. The critical current is seen to be characterisitc of the junction and independent of $\lambda$. The incoming current from $B_1$ splits up equally and flows into $B_2$ and $B_3$. Note that there is an enhancement of the critical current by the factor $\sqrt{2}$.

2. *The asymmetrical case with* $\alpha = 0$. Here we can simply carry out the correlations between $\hat{P}^{(2)}_{\lambda 2}$ and $K^{(2)}_{a,b}$ as for the two-branch circuit to obtain the Josephson equation

$$j = j_0 \sin\lambda_2 \tag{52}$$

where

$$j = j^{(3)}_{\lambda 2p} = \frac{q}{m}p, \qquad j_0 = \frac{q}{m}\hbar\, b > 0 \qquad \text{on assuming} \quad b > 0 \tag{53}$$

The incoming current $I$ in $B_1$ goes straight through to $B_2$ with no current going into $B_3$.

3. *The other extreme case with* $\alpha = 1$. The situation is the same except with $B_2$, $B_3$ interchanged, i.e., the current flows in from $B_1$ into $B_3$ without branching into $B_2$.

However, this is not the end of the story. To complete the theory, a superselection rule has to be introduced. In the two asymmetric cases, which are essentially the same as two-branch circuits, we have already established the necessary superselection rule (Wan and Fountain, 1996). We shall present the analysis for the superselection rule for the symmetrical case in what follows.

## 4.5. Superselection Rules for the Symmetrical Case

Recall that assumption (PA4) requires the current $I$ to determine the value of $\lambda$ so as to single out a current operator $\hat{J}^{(3)}_{\lambda}$. Since the value of $\lambda$ is uniquely determined by $j = I$ only over a range of $\pi$, assumption (PA4) restricts $\lambda$ to the range $[-\frac{1}{2}\pi, 0) \cup (0, \frac{1}{2}\pi]$, say. This restriction also helps to determine the value of $a$ and hence the kinetic energy operator $\hat{K}^{(3)}_{a,b}$. We conclude that our physical system with a given Josephson junction and a current $I$ determines a pair $a,b$ since:

1. $b$ is a characteristic of the junction
2. $\lambda$ is fixed by $I = j = \underline{j}_0 \sin \lambda$
3. $a$ is fixed by $a = -\sqrt{2b} \cos \lambda$

It follows that our system determines a unique momentum $\hat{P}^{(3)}_{\lambda}$, a unique supercurrent $\hat{J}^{(3)}_{\lambda}$, and a unique kinetic energy operator $\hat{K}^{(3)}_{a,b}$.

The analysis set out for the two-branch circuit (Wan and Fountain, 1996) and for the one-branch circuit discussed in the preceding section which establishes a superselection rule applies here. The result is the same as before in that a continuous superselection rule exists parametrized by the current $I$. This superselection rule again reduces each supersector to one dimension. Consequently the appropriate supercurrent operator and the Hamiltonian are the restrictions $\hat{J}^{(3)}_{\lambda}$ and $\hat{K}^{(3)}_{\lambda}$ to the corresponding one-dimensional subspaces.

To sum up, we now have three descriptions of the circuit, one symmetrical and two asymmetrical. It may seem natural to assume that the former is the correct description of the circuit. However, the analysis presented in the next subsection reveals that this may not necessarily be the case.

## 4.6. Condensate in a Pure or in a Mixed State

An experiment to examine how a supercurrent fed into $B_1$ will flow down the circuit could have the following three possible results:

1. The current from $B_1$ flows entirely into $B_2$ with no current in $B_3$. This means that the condensate is in a pure state described by $\varphi^{(3)}_{\lambda 2p}$.
2. The current from $B_1$ flows entirely into $B_3$ with no current in $B_2$. This means that the condensate is in a pure state described by $\varphi^{(3)}_{\lambda 3p}$.
3. The current from $B_1$ splits up and flows equally into $B_2$ and $B_3$ with enhancement of the critical current as mentioned before.

The first two cases above are unambiguous. Suppose that an experiment confirms case 3 above, namely that the current from $B_1$ splits up and flows equally into $B_2$ and $B_3$. The question now is whether we jump to the following conclusions:

1. The states are pure corresponding to $\varphi^{(3)}_{\lambda p}$ with a superselection rule.
2. The supercurrent operator and the Hamiltonian of the system are described by the restrictions of $J^{(3)}_{\lambda}$ and $\mathcal{H}^{(3)}_{a,b}$ onto the one-dimensional supersectors defined by $\varphi^{(3)}_{\lambda p}$.

In a previous paper (Wan and Fountain, 1996) it is argued that we should not jump to the above conclusions. Instead the state should really be a mixture of $\varphi^{(3)}_{\lambda 2p}$ and $\varphi^{(3)}_{\lambda 3p}$ rather then the pure state $\varphi^{(3)}_{\lambda p}$. In other words, the condensate consists of a mixture of two parts; one part corresponds to $\varphi^{(3)}_{\lambda 2p}$ and the other part corresponds to $\varphi^{(3)}_{\lambda 3p}$. This conclusion is motivated by physical considerations, details of which are given in Wan and Fountain (1996). A mixture of $\varphi^{(3)}_{\lambda 2p}$ and $\varphi^{(3)}_{\lambda 3p}$ means that the incoming current in $B_1$ consists of two components with one component flowing into $B_2$ and the other component flowing into $B_3$. In other words, the Cooper pairs in $B_1$ are divided into two groups described separately by $\varphi^{(3)}_{\lambda 2p}$ and $\varphi^{(3)}_{\lambda 3p}$; each group forms a current component. The two components never meet up again to form an interference circuit. Because of the intrinsic global nature of the condensate in dc effects the kind of delay-choice experiments described in Wheeler and Zurek (1983) are not appropriate here. It follows that the current in $B_1$ splitting up into $B_2$ and $B_3$ is not accompanied by each Cooper pair having to "split up" into $B_2$ and $B_3$. If each Cooper pair were to split up and then never meet up again we would be confronted with a de Broglie-type paradox (Selleri and Tarozzi, 1981; Wan and McLean, 1984), namely we are faced with the problem of not knowing what happens to the electron pair after the splitting, e.g., in which branch the electrons are and so on. The situation would be quite different if $B_2$ and $B_3$ were to meet up to form an interference circuit (Wan and Fountain, 1996; Wollman *et al.*, 1993; Harrison and Wan, 1997).

Finally our model of the three-branch circuit offers a chance to test by experiment whether the condensate is in a pure state or in a mixed state. According to the argument just presented, a confirmation of current spliting into $B_2$ and $B_3$ would contradict the above statement. If the condensate is necessarily in a pure state, then an experiment should confirm the counterintuitive result that current from $B_1$ flows entirely into either $B_2$ or $B_3$ without splitting.

As far as an experimental test is concerned, it is probably easier to employ the continuous Y-circuit below.

### 4.7. Continuous Y-Circuit

A simple but nontrivial configuration not included in the preceding discussions is that of a Y-circuit where the three branches join up at the branch point to eliminate the Josephson junction. For such a continuous circuit configuration one may be tempted to adopt the seemingly natural continuity conditions on the three branches, i.e., $\phi_{20} = \phi_{10} = \phi_{30}$. However, these conditions are inappropriate since they contradict the general boundary conditions given by equations (31) and (32) for maximal symmetric extensions for the composite momentum operator. We shall again confine our attention to three cases.

1. Symmetrical case with boundary conditions $\alpha = 2^{-1/2}$, $\lambda_2 = \lambda_3$ for the momentum operator. We assume continuity of the phase at the branch point, i.e., $\lambda_2 = \lambda_3 = 0$, to distinguish our present, continuous circuit from the previous, discontinuous one. The resulting composite momentum, denoted by $\hat{P}^{(3)}_{\lambda=0}$, admits eigenfunctions

$$\varphi^{(3)}_{\lambda=0,p} = e^{ipx_1} \oplus \frac{1}{\sqrt{2}} e^{ipx_2} \oplus \frac{1}{\sqrt{2}} e^{ipx_3} \tag{54}$$

The supercurrent operator is $\hat{J}^{(3)}_{\lambda=0} = (q/m)\, \hat{P}^{(3)}_{\lambda=0}$.

To determine the kinetic energy, we impose a further boundary condition (Appendix D):

$$\Phi'_{10} = \frac{1}{\sqrt{2}} (\Phi'_{20} + \Phi'_{30}) \tag{55}$$

which, together with the condition

$$\Phi_{10} = \frac{1}{\sqrt{2}}(\Phi_{20} + \Phi_{30}) \tag{56}$$

already implied by the continuity of the phase at the branch point, determines a kinetic energy operator $\hat{K}^{(3)}_{\lambda=0}$ admitting $\varphi^{(3)}_{\lambda=0,p}$ as an eigenfunction (Appendix E).

We have an interesting situation where the wave function still suffers a discontinuity at the branch point despite the continuity of the circuit at the branch point. The incoming current in $B_1$ splits up equally and flows into $B_2$ and $B_3$.

2. Asymmetrical case with boundary conditions $\alpha = 0$, $\lambda_2 = 0$. This amounts to continuity between branches $B_1$ and $B_2$ while ignoring $B_3$, i.e.,

$$\phi_{10} = \phi_{20}, \qquad \phi_{30} = 0 \tag{57}$$

The resulting composite momentum, denoted by $\hat{P}^{(3)}_{\lambda_2=0}$, is equal to $\hat{p} \oplus \hat{p}_3$, where $\hat{p}$ is the usual momentum in $L^2(\mathbb{R}) = \mathcal{H}_1 \oplus \mathcal{H}_2$. The momentum $\hat{P}^{(3)}_{\lambda_2=0}$ admits eigenfunctions of the form

$$\varphi^{(3)}_{\lambda_2=0,p} = e^{ipx_1} \oplus e^{ipx_2} \oplus O_3 \tag{58}$$

The supercurrent operator is $\hat{J}^{(3)}_{\lambda_2=0} = (q/m)\, \hat{P}^{(3)}_{\lambda_2=0}$. The kinetic energy can easily be established. Let $\hat{K} = \hat{p}^2/2m$ be the usual kinetic energy operator in $L^2(\mathbb{R}) = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then we can take the composite kinetic energy as

$$\hat{K}^{(3)}_{\lambda_2=0} = \hat{K} \oplus \hat{K}_3, \qquad \text{where} \qquad \hat{K}_3 = \frac{1}{2m} \hat{p}_3^{\dagger} \hat{p}_3 \tag{59}$$

Physically this describes an incoming current in $B_1$ going straight through to $B_2$ with no current flowing into $B_3$.

3. Asymmetric case with with the boundary conditions $\alpha = 0$, $\lambda_3 = 0$. Everything is the same as case except with $B_2$ and $B_3$ interchanged.

The problem of whether the condensate is in a pure or a mixed state discussed in the preceding subsection applies here.

## 5. CONCLUDING REMARKS

It is now clear that our method of quantization by parts can be applied to multibranch circuits. Examples which are seen to be easily soluble are a four-branch circuit consisting of two leads connecting a thick superconducting ring, a five-branch circuit consisting of two leads connecting a thick, superconducting ring interrupted by a Josephson junction (the so-called rf-SQUID configuration), and a six-branch circuit consisting of two leads connecting a thick, superconducting ring interrupted by two Josephson junctions (the so-called dc-SQUID configuration). Work is in progress to investigate a series of other circuit configurations.

## APPENDIX A. KINETIC ENERGY OPERATOR FOR TSCR WITH A JJ

First, symmetry consideration apparently produces two possible alternatives:

$$\underline{\tilde{\alpha}} = \delta\underline{\alpha}, \quad \underline{\tilde{\beta}} = \delta'\underline{\beta} \quad \textit{or} \quad \underline{\tilde{\alpha}} = \delta\underline{\beta}, \quad \underline{\tilde{\beta}} = \delta'\underline{\alpha} \quad \textit{for some} \quad \delta, \delta' \in \mathbb{C}$$

Consider the first alternative. Setting the proportionality constants to 1, we get

$$\alpha'_- = -\alpha'_+, \quad \alpha_- = \alpha_+ \qquad \text{and} \qquad \beta'_- = -\beta'_+, \quad \beta_- = \beta_+ \tag{60}$$

Using equations (60), we can write conditions (BC1) as

$$\alpha'_-(\phi'_- + \phi'_+) = \alpha_-(\phi_- - \phi_+) \tag{61}$$

$$\beta'_-(\phi'_- + \phi'_+) = \beta_-(\phi_- - \phi_+) \tag{62}$$

$$\Rightarrow \frac{\alpha'_-}{\beta'_-} = \frac{\alpha_-}{\beta_-} \tag{63}$$

This result is insufficient to contradict (C1). However, using equation (60) again, we can write conditions (BC1) as

$$\alpha'_+(\phi'_+ + \phi'_-) = \alpha_-(\phi_+ - \phi_-) \tag{64}$$

$$\beta'_+(\phi'_+ + \phi'_-) = \beta_-(\phi_+ - \phi_-) \tag{65}$$

$$\Rightarrow \frac{\alpha'_+}{\beta'_+} = \frac{\alpha_-}{\beta_-} \tag{66}$$

We can continue this process until we have gone through all the combinations. As a result of this, (C1) is contradicted.

Intuitively one can easily check that many of the above equations are immediately contradicted by the best-known self-adjoint extension, i.e., the extension defined by the periodic boundary conditions $\phi_- = \phi_+$, $\phi'_- = \phi'_+$. We conclude that the second alternative is the correct one.

Now, we can divide the second alternative into a number of scenarios:

1. Scenario 1 with none of the parameters vanishing.

(a) Case 1 with $\alpha_-^{'2} - \alpha_+^{'2} \neq 0$. The boundary conditions (20) and (21) can again be written in the form of (BC2) with

$$a = \frac{\alpha_+\alpha'_+ + \alpha_-\alpha'_-}{\alpha_-^{'2} - \alpha_+^{'2}}, \qquad b = -\frac{\alpha_+\alpha'_- + \alpha'_+\alpha_-}{\alpha_-^{'2} - \alpha_+^{'2}} \tag{67}$$

Conditions (C2) imply that $a$, $b$ are real.

(b) Case 2 with $\alpha_-^{\prime 2} - \alpha_+^{\prime 2} = 0$. When $\alpha'_- = \alpha'_+$ conditions (20) and (21) reduce to $\phi_- = \phi_+$, which would correlate with $\hat{p}_\lambda$, $\lambda = 0$, and should be rejected. When $\alpha'_- = \alpha'_+$ conditions (20) and (21) reduce to $\phi_- = -\phi_+$; this seems to correlate with $\hat{p}_\lambda$, $\lambda = \pi$. But substituting this into (BC1), we get $\phi'_- = -\phi'_+$, which is violated by the eigenfunctions of $\hat{p}_{\lambda=\pi}$. This inconsistency is ground for rejection of $\hat{p}_{\lambda=\pi}$, and hence this case with $\alpha_-^{\prime 2} - \alpha_+^{\prime 2} = 0$.

2. Scenario 2 with one of the parameters vanishing.

(a) Case 1 with $\alpha'_+ = 0$, $\alpha'_- = 1$, $\alpha_- = a \in \mathbb{R}$, $\alpha_+ = -b \in \mathbb{R}$. This reduces the boundary conditions (20) and (21) to the form of (BC2)

$$\phi'_- = a\phi_- + b\phi_+, \qquad \phi'_+ = -a\phi_+ - b\phi_-, \qquad a, b \in \mathbb{R} \tag{68}$$

(b) Case 2 with $\alpha'_- = 0$, $\alpha'_+ = 1$, $\alpha_- = b \in \mathbb{R}$, $\alpha_+ = -a \in \mathbb{R}$. This also reduces the boundary conditions (20) and (21) to the form of (BC2)

$$\phi'_- = a\phi_- + b\phi_+, \qquad \phi'_+ = -a\phi_+ - b\phi_-, \qquad a, b \in \mathbb{R} \tag{69}$$

(c) Case 3 with $\alpha_- = 0$, $\alpha_+ = 1$. Condition (C2) implies that $\alpha'_+$, $\alpha'_-$ are real and the boundary conditions (20) and (21) reduce to

$$\phi_- = \alpha'_-\phi'_- - \alpha'_+\phi'_+ \tag{70}$$

$$\phi_+ = -\alpha'_+\,\phi'_- - \alpha'_-\,\phi'_+ \tag{71}$$

which can again be reduced to the form of (BC2) with

$$a = \frac{-\alpha'_+}{\alpha_+^{\prime 2} - \alpha_-^{\prime 2}}, \qquad b = \frac{\alpha'_-}{\alpha_+^{\prime 2} - \alpha_-^{\prime 2}} \tag{72}$$

Both $a$ and $b$ are real.

(d) Case 4 with $\alpha_+ = 0, \alpha_- = 1$. Condition (C2) implies that $\alpha'_+, \alpha'_-$ are real and the boundary conditions (20) and (21) reduce to

$$\phi_- = \alpha'_- \phi'_- - \alpha'_+ \phi'_+ \tag{73}$$

$$\phi_+ = -\alpha'_+ \phi'_- - \alpha'_- \phi'_+ \tag{74}$$

which can be reduced to the form of (BC2) with

$$a = \frac{-\alpha'_+}{\alpha'^2_+ - \alpha'^2_-}, \qquad b = \frac{\alpha'_-}{\alpha'^2_+ - \alpha'^2_-} \tag{75}$$

Both $a$ and $b$ are real.

3. Scenario 3 with two of the parameters vanishing.

(a) Case 1 with $\alpha_- = \alpha'_- = 0$ or $\alpha_+ = \alpha'_+ = 0$. With $\alpha_- = \alpha'_- = 0$ conditions (20) and (21) reduce to what are known as separated conditions (Hudson and Pym, 1980):

$$\alpha'_+ \phi'_+ - \alpha_+ \phi_+ = 0, \qquad \alpha'_+ \phi'_- + \alpha_+ \phi_- = 0 \tag{76}$$

which are violated by the eigenfunctions of $\hat{p}_\lambda$, and hence rejected. The same applies with $\alpha_+ = \alpha'_+ = 0$.

(b) Case 2 with $\alpha_- = \alpha_+ = 0$ or $\alpha'_+ = \alpha'_- = 0$. With $\alpha_- = \alpha_+ = 0$ conditions (20) and (21) reduce to $(\phi'_-)^2 = (\phi'_+)^2$, which is violated by the eigenfunctions of $\hat{p}_\lambda$, $\lambda \neq 0, \pi$, and hence rejected. With $\alpha'_+ = \alpha'_- = 0$ we obtain $(\phi_-)^2 = (\phi_+)^2$, which should be rejected for the same reason.

(c) Case 3 with $\alpha_- = \alpha'_+ = 0$ or $\alpha_+ = \alpha_- = 0$. With $\alpha_- = \alpha'_+ = 0$ we get $\phi_- \phi'_- = -\phi_+ \phi'_+$, which is violated by the eigenfunctions of $\hat{p}_\lambda$, $\lambda \neq 0$, $\pi$, and hence rejected. The same applies if $\alpha_+ = \alpha_- = 0$.

4. Scenario 4 with three of the parameters vanishing. Conditions (20) and (21) would lead the vanishing of $\phi$ and $\phi'$ at the junction, and should hence be rejected.

## APPENDIX B. MOMENTUM OPERATORS FOR THE THREE-BRANCH CIRCUIT

The adjoint of $\hat{P}_0^{(3)}$ in $\mathcal{H}^{(3)}$ is

$$\hat{P}_0^{(3)\dagger} = \hat{p}_1^\dagger \oplus \hat{p}_2^\dagger \oplus \hat{p}_3^\dagger \tag{77}$$

The domain of the adjoint operator is

$$\mathcal{D}(\hat{P}_0^{(3)\dagger}) = \{\phi = \phi_1 \oplus \phi_2 \oplus \phi_3 \in \mathcal{H}^{(3)}: \phi_l \in AC(B_l), d\phi_l/dx_l \in \mathcal{H}_l\} \tag{78}$$

where $AC(B_l)$ is the set of absolutely continuous functions on $B_l$. The deficiency spaces of $\hat{P}_0^{(3)}$ are

$$\mathcal{N}_{\pm} = \{\phi \in \mathcal{D}(\hat{P}_0^{(3)\dagger}): (\hat{P}_0^{(3)\dagger} \mp i)\phi = 0\} \tag{79}$$

$\mathcal{N}_-$ is one-dimensional and is spanned by an orthonormal basis consisting of the following element:

$$f_- = e^{x_1} \oplus 0 \oplus 0, \qquad x_1 \in (-\infty, 0) \tag{80}$$

$\mathcal{N}_+$ is two-dimensional and is spanned by the orthonormal basis

$$g_2 = 0 \oplus e^{-x_2} \oplus 0, \qquad g_3 = 0 \oplus 0 \oplus e^{-x_3}, \qquad x_2, x_3 \in (0, \infty) \tag{81}$$

According to the second von Neumann formula (Weidman, 1980) an operator $\hat{P}^{(3)}$ in $\mathcal{H}^{(3)}$ is a closed symmetric extension of $\hat{P}_0^{(3)}$ if and only if there are closed subspaces $\mathcal{F}_+$ of $\mathcal{N}_+$ and $\mathcal{F}_-$ of $\mathcal{N}_-$ and an isometric mapping $\hat{V}$ of $\mathcal{F}_+$ onto $\mathcal{F}_-$ such that

$$\mathcal{D}(\hat{P}^{(3)}) = \{\phi \in \mathcal{D}(\hat{P}_0^{(3)})\} + \{g_+ + \hat{V}g_+: g_+ \in F_+\} \tag{82}$$

and

$$\hat{P}^{(3)}(\phi + g_+ + \hat{V}g_+) = \hat{P}_0^{(3)}\phi + ig_+ - i\,\hat{V}g_+ \tag{83}$$

Clearly $\mathcal{F}_- = \mathcal{N}_-$ and it follows that all closed symmetric extensions of $\hat{P}_0^{(3)}$ are maximal symmetric.

Now, $\mathcal{F}_+$ must be one-dimensional and every one-dimensional closed subspace of $\mathcal{N}_+$ is spanned by a normalized function of the form

$$g_+ = \sqrt{1 - \alpha^2}\, e^{i\lambda_2'}\, g_2 + \alpha e^{i\lambda_3'}\, g_3 \tag{84}$$

$$= 0 \oplus \sqrt{1 - \alpha^2} e^{i\lambda_2}\, e^{-x_2} \oplus \alpha e^{i\lambda_3'}\, e^{-x_3}, \tag{85}$$

$$\alpha \in [0, 1]; \quad \lambda_2', \lambda_3' \in (-\pi, \pi]$$

Hence all relevant isometric mappings from $\mathcal{F}_+$ onto $\mathcal{F}_-$ are given by

$$\hat{V}g_+ = e^{i\theta} f_-, \qquad \theta \in (-\pi, \pi] \tag{86}$$

It follows that the domain of $\hat{P}^{(3)}$ must consist of functions of the form

$$\Phi = \phi + g_+ + \hat{V}g_+ = \phi + c_+(g_+ + e^{i\theta} f_-), \qquad c_+ \in \mathbb{C} \tag{87}$$

$$= \phi + c_+(e^{i\theta} e^{x_1} \oplus \sqrt{1 - \alpha^2} e^{\,i\lambda_2'}\, e^{-x_2} \oplus \alpha e^{\,i\lambda_3'}\, e^{-x_3}) \tag{88}$$

Observe that $\Phi$ is of the form $\Phi = \Phi_1 \oplus \Phi_2 \oplus \Phi_3$, where

$$\begin{aligned} \Phi_1 &= \phi_1 + c_+ e^{i\theta}\, e^{x_1} \\ \Phi_2 &= \phi_2 + c_+\sqrt{1 - \alpha^2}\, e^{i\lambda_2'}\, e^{-x_2} \\ \Phi_3 &= \phi_3 + c_+ \alpha e^{i\lambda_3'}\, e^{-x_3} \end{aligned} \tag{89}$$

These give rise to the following boundary conditions:

$$\Phi_{20} = \sqrt{1-\alpha^2}\, e^{i\lambda_2}\Phi_{10},$$
$$\Phi_{30} = \alpha e^{i\lambda_3}\Phi_{10}, \qquad \text{where} \quad \lambda_2, \lambda_2 \in (-\pi, \pi] \tag{90}$$

In other words we have boundary conditions specified by three real parameters

$$\Phi_{20} = \sqrt{1-\alpha^2}\, e^{i\lambda_2}\Phi_{10} \tag{91}$$

$$\Phi_{30} = \alpha e^{i\lambda_3}\Phi_{10} \tag{92}$$

The generalized eigenfunctions are of the form

$$\varphi^{(3)}_{p\alpha\lambda_2\lambda_3} = e^{\bar{\imath}px_1} \oplus \sqrt{1-\alpha^2}\, e^{i\lambda_2}\, e^{\bar{\imath}px_2} \oplus \alpha e^{i\lambda_3} e^{\bar{\imath}px_3}, \qquad \alpha \in (-\pi, \pi] \tag{93}$$

with generalized eigenvalues $p$. These functions for each chosen set of values of $\alpha$, $\lambda_2$, $\lambda_3$ do not form a complete set in $\mathcal{H}^{(3)}$. They are orthogonal, namely

$$\langle \varphi^{(3)}_{p\alpha\lambda_2\lambda_3} | \varphi^{(3)}_{p'\alpha\lambda_2\lambda_3} \rangle = 2\pi\hbar\delta(p-p') \tag{94}$$

where

$$\langle \varphi^{(3)}_{p\alpha\lambda_2\lambda_3} | \varphi^{(3)}_{p'\alpha\lambda_2\lambda_3} \rangle = \int_{-\infty}^{0} (e^{\bar{\imath}px_1})^* \, (e^{ip'x_1})\, dx_1 \tag{95}$$

$$+ \int_0^{\infty} (\sqrt{1-\alpha^2}\, e^{i\lambda_2} e^{\bar{\imath}px_2})^* \tag{96}$$

$$(\sqrt{1-\alpha^2} e^{i\lambda_2} e^{\bar{\imath}p'x_2})\, dx_2 \tag{97}$$

$$+ \int_0^{\infty} (\alpha e^{i\lambda_3} e^{\bar{\imath}px_3})^* \, (\alpha e^{i\lambda_3} e^{\bar{\imath}p'x_3})\, dx_3$$

When integrating in $p$ we get

$$\int_{-\infty}^{\infty} (e^{\bar{\imath}px_1})^* \, (e^{\bar{\imath}px_1'})\, dp \tag{98}$$

$$+ \int_{-\infty}^{\infty} (\sqrt{1-\alpha^2}\, e^{i\lambda_2} e^{\bar{\imath}px_2})^* \, (\sqrt{1-\alpha^2}\, e^{i\lambda_2} e^{\bar{\imath}px_2'})\, dp \tag{99}$$

$$+ \int_{-\infty}^{\infty} (\alpha e^{i\lambda_3}\, e^{\bar{\imath}px_3})^* \, (\alpha e^{i\lambda_3} e^{\bar{\imath}px_3'})\, dp$$

$$= 2\pi\hbar(\delta(x_1 - x_1') + (1-\alpha^2)\, \delta(x_2 - x_2') + \alpha^2\, \delta(x_3 - x_3')) \tag{100}$$

showing that these functions do not form a complete set on account of factors in $\alpha$. To be complete in $\mathcal{H}^{(3)}$, a set of functions has to be complete in $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$. This is as expected for a maximal symmetric operator. For the extreme cases $\alpha = 0$ and $\alpha = 1$, the set $\{\varphi^{(3)}_{p\alpha\lambda_2\lambda_3}\}$ can be said to be complete in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}_1 \oplus \mathcal{H}_3$, respectively. Clearly there are too many possible momentum operators. We have to proceed to the correlations stage in order to be in a position to make a choice. In other words, we have to quantize other related observables first and try to build up a correlation. As will be seen later, this correlation enables us to establish a choice of momentum operator. In the present case the other two cognate observables are the supercurrent and the kinetic energy.

## APPENDIX C. EIGENFUNCTIONS OF MAXIMAL SYMMETRIC OPERATORS

The incompleteness of the eigenfunctions of a maximal symmetric operator which is not self-adjoint can be seen more clearly in the following example. Consider a momentum operator $\hat{p}_\lambda$ for the interval $(0, 2\pi)$ and the momentum operator $\hat{p}_+$ for the half-line $\mathbb{R}^+$. Here $\hat{p}_\lambda$ is self-adjoint in $L^2(0, 2\pi)$ with eigenfunctions $\psi_{\lambda,n}$, and $\hat{p}_+$ is maximal symmetric in $L^2(\mathbb{R}^+)$. The operator $\hat{p}_\lambda \oplus \hat{p}_+$ in $L^2(0, 2\pi) \oplus L^2(\mathbb{R}^+)$ is maximal symmetric, but not self-adjoint. The eigenfunctions of $\hat{p}_\lambda \oplus \hat{p}_+$ are of the form $\psi_{\lambda,n} \oplus 0$, which clearly form an orthogonal and incomplete set in $L^2(0, 2\pi) \oplus L^2(\mathbb{R}^+)$. This has immediate physical implications, as seen in the discussion for the three-branch circuit.

## APPENDIX D. KINETIC ENERGY OPERATORS FOR THE THREE-BRANCH CIRCUIT

The partially quantized operators possess self-adjoint extensions. For example, $\hat{K}_{30}$ in $\mathcal{H}_3$ is known to have a one-parameter family of self-adjoint extensions (Reed and Simon, 1975). An obvious self-adjoint extension is $\hat{K}_3 = (1/2m)\,\hat{p}_3^\dagger \hat{p}_3$ since this is compatible with $\hat{p}_3$ in that the domains $\mathcal{D}(\hat{K}_3)$ and $\mathcal{D}(\hat{p}_3)$ both consist of functions vanishing at the origin.

For composite quantization consider the direct sum

$$\hat{K}_0^{(3)} = \hat{K}_{01} \oplus \hat{K}_{02} \oplus \hat{K}_{03} \qquad \text{defined on}$$

$$\mathcal{D}(\hat{K}_0^{(3)}) = \mathcal{D}(\hat{K}_{01}) \oplus \mathcal{D}(\hat{K}_{02}) \oplus \mathcal{D}(\hat{K}_{03}) \tag{101}$$

### D1. The Neumann Formula

Let $\overline{\Delta}_0^{(3)}$ be the closure of $(2m/\hbar^2)\,\hat{K}_0^{(3)}$; then we have $\overline{\Delta}_0^{(3)} = \overline{\Delta}_{01}^- \oplus \overline{\Delta}_{02}^+ \oplus \overline{\Delta}_{03}^+$, where $\overline{\Delta}_{01}^-$, $\overline{\Delta}_{02}^+$, and $\overline{\Delta}_{03}^+$ are the closures of $(2m/\hbar^2)\hat{K}_{01}$,

$(2m/\hbar^2)\hat{K}_{02}^{+}$, and $(2m/\hbar^2)\hat{K}_{03}^{+}$, respectively. Let $\mathcal{N}_+$, $\mathcal{N}_-$ be the deficiency subspaces of $\overline{\Delta}_0^{(3)}$, i.e.,

$$\mathcal{N}_+ = \{\phi \in \mathcal{H}^{(3)}: (\overline{\Delta}_0^{(3)\dagger} - i)\phi = 0\}, \qquad \mathcal{N}_- = \{\phi \in \mathcal{H}^{(3)}: (\overline{\Delta}_0^{(3)\dagger} + i)\phi = 0\}$$

If $\overline{\Delta}_s^{(3)}$ denotes a self-adjoint extension of $\overline{\Delta}_0^{(3)}$, then, by the second von Neumann formula (Weidman, 1980; Blank *et al.*, 1994), the domain of $\overline{\Delta}_s^{(3)}$ is

$$\mathcal{D}(\overline{\Delta}_s^{(3)}) = \mathcal{D}(\overline{\Delta}_0^{(3)}) + \{\varphi + \hat{V}\varphi: \varphi \in \mathcal{N}_+\} \tag{102}$$

where $\hat{V}$ is a unitary mapping of $\mathcal{N}_+$ onto $\mathcal{N}_-$, and on $\mathcal{D}(\overline{\Delta}_s^{(3)})$ we have

$$\overline{\Delta}_s^{(3)}\,\Phi = \overline{\Delta}_0^{(3)}\,\phi + i\varphi - i\hat{V}\varphi, \qquad \Phi = \phi + \varphi + \hat{V}\varphi \in \mathcal{D}(\overline{\Delta}_s^{(3)}) \tag{103}$$

The deficiency spaces of $\hat{K}_0^{(3)}$ are

$$\mathcal{N}_\pm = \{\phi \in \mathcal{D}(\hat{P}_0^{(3)\dagger}): (\hat{K}_0^{(3)\dagger} \mp i)\phi = 0\} \tag{104}$$

$\mathcal{N}_-$ is three-dimensional and is spanned by an orthonormal basis consisting of the following elements

$$f_1 = ce^{\rho^* x_1} \oplus O_2 \oplus O_3, \qquad c = 2^{1/4}, \quad \rho = e^{-i\pi/4} \tag{105}$$

$$f_2 = O_1 \oplus ce^{-\rho^* x_2} \oplus O_3 \tag{106}$$

$$f_3 = O_1 \oplus O_2 \oplus ce^{-\rho^* x_3} \tag{107}$$

$\mathcal{N}_+$ is three-dimensional and is spanned by the orthonormal basis

$$g_1 = ce^{\rho x_1} \oplus O_2 \oplus O_3 \tag{108}$$

$$g_2 = O_1 \oplus ce^{-\rho x_2} \oplus O_3 \tag{109}$$

$$g_3 = O_1 \oplus O_2 \oplus ce^{-\rho x_3} \tag{110}$$

Let

$$\varphi = \sum_k c_k g_k, \; \hat{V} g_k = \sum_j u_{jk} f_j, \qquad \text{where} \qquad j, k = 1, 2, 3$$

Note that $\rho = (1 - i)/\sqrt{2}$ and $\rho^*\rho = 1$, $\rho^2 = -i$, $\rho^{*2} = i$. Then

$$\Phi = \phi + \sum_k c_k \left( g_k + \sum_j u_{jk} f_j \right), \qquad \phi \in \mathcal{D}(\overline{\Delta}_0^{(3)}) \tag{111}$$

The $3 \times 3$ unitary matrix $(u_{jk})$ means that we have a nine-parameter family of self-adjoint extensions.

## D2. Permutation Invariant with Respect to $B_2$, $B_3$

Let us consider extension operators with domains containing functions $\Phi = \Phi_1 \oplus \Phi_2 \oplus \Phi_3$ all satisfying the following three boundary conditions at the junction $x_i = 0$:

$$\Phi_{30} = \Phi_{20} \tag{112}$$

$$\Phi'_{10} = a\Phi_{10} + b(\Phi_{20} + \Phi_{30}) \tag{113}$$

$$\Phi'_{20} + \Phi'_{30} = -a(\Phi_{20} + \Phi_{30}) - b2\Phi_{10} \tag{114}$$

where the prime indicates a derivative with respect to an appropriate position variable $x_i$ and subscript 0 signifies the value taken at the junction $x_i = 0$. The first boundary condition imposes the following relations between the matrix elements:

$$u_{12} = u_{13}, \qquad u_{21} = u_{31}, \qquad u_{22} = u_{33}, \qquad u_{23} = u_{32} = 1 + u_{22} \tag{115}$$

These relations in turn give rise to the following expressions for the boundary values of the wave function and its derivatives:

$$\Phi_{10} = c_1(1 + u_{11}) + (c_2 + c_3)u_{12} \tag{116}$$

$$\Phi_{20} = c_1 u_{21} + (c_2 + c_3)(1 + u_{22}) \tag{117}$$

$$\Phi_{30} = \Phi_{20} \tag{118}$$

$$\Phi'_{10} = c_1(\rho + \rho^* u_{11}) + (c_2 + c_3)\rho^* u_{12} \tag{119}$$

$$\Phi'_{20} = c_1(-\rho^* u_{21}) - c_2(\rho + \rho^* u_{22}) - c_3\rho^*(1 + u_{22}) \tag{120}$$

$$\Phi'_{30} = c_1(-\rho^* u_{21}) - c_2\rho^*(1 + u_{22}) - c_3(\rho + \rho^* u_{22}) \tag{121}$$

Note that $\Phi'_{30} = \Phi'_{20}$ only when $c_2 = c_3$.

Next, the second and the third boundary conditions together with the above explicit expressions from the von Neumann formula give

$$\rho + \rho^* u_{11} = a(1 + u_{11}) + b2u_{21} \tag{122}$$

$$\rho^* u_{12} = au_{12} + b2(1 + u_{22}) \tag{123}$$

$$\rho^* u_{21} = au_{21} + b(1 + u_{11}) \tag{124}$$

$$\rho + \rho^*(1 + 2u_{22}) = a2(1 + u_{22}) + b2u_{12} \tag{125}$$

These equations are consistent if the matrix elements are suitably related. Let us assume that the matrix elements are related by

$$u_{12} = u_{21}, \qquad 1 + u_{11} = 2(1 + u_{22}) \tag{126}$$

This is equivalent to the requirement that equations (122) and (125) are the same and equations (123) and (124) are the same. Then the above four equations are reduced to two

$$\rho + \rho^* u_{11} = a(1 + u_{11}) + b2u_{12} \tag{127}$$

$$\rho^* u_{12} = au_{12} + b(1 + u_{11}) \tag{128}$$

leading to the following explicit relations:

$$u_{11} = -\frac{-1 + \sqrt{2}a - a^2 + 2b^2}{i - 2a\rho^* + a^2 - 2b^2} \tag{129}$$

$$u_{12} = \frac{b}{(\rho^* - a)} \frac{i(1 - \sqrt{2}a) - 1}{(i - 2a\rho^* + a^2 - 2b^2)} \tag{130}$$

$$a = \rho^* - \frac{i\sqrt{2}(1 + u_{11})}{(1 + u_{11})^2 - 2u_{12}^2} \tag{131}$$

$$b = \frac{i\sqrt{2}u_{12}}{(1 + u_{11})^2 - 2u_{12}^2} \tag{132}$$

It follows that $a, b$ determine a matrix $(u_{jk})$ and vice versa. To sum up we have:

1. The matrix $(u_{jk})$ of the form

$$\begin{pmatrix} 1 + 2u & v & v \\ v & u & 1 + u \\ v & 1 + u & u \end{pmatrix} \tag{133}$$

which can be shown to be unitary if the complex numbers $u$, $v$ satisfy

$$u + u^* + 2|u|^2 + |v|^2 = 0 \tag{134}$$

$$v + v^* + 2u^*v + 2v^*u = 0 \tag{135}$$

2. The matrix $(u_{jk})$ is determined by two independent real parameters $a$, $b$. Moreover, $a$, $b$ determines a self-adjoint extension $\overline{\Delta}_s^{(3)}$ provided the matrix $(u_{jk})$ is unitary.

## APPENDIX E. CONTINUOUS Y-CIRCUIT

First the general momentum boundary conditions with $\alpha = 1/\sqrt{2}$, $\lambda_2 = \lambda_3 = 0$ imply

$$\Phi_{10} = \frac{1}{\sqrt{2}}(\Phi_{20} + \Phi_{30}) \tag{136}$$

For the composite kinetic energy we impose a further boundary condition

$$\Phi'_{10} = \frac{1}{\sqrt{2}}(\Phi'_{20} + \Phi'_{30}) \tag{137}$$

Substitute these equations into the expressions obtained from the von Neumann formula to get

$$1 + u_{11} = \sqrt{2}u_{12} \tag{138}$$

$$-i + u_{11} = -\sqrt{2}u_{12} \tag{139}$$

These matrix elements correspond to

$$u = \frac{1}{4}(i - 3), \qquad v = \frac{1}{2\sqrt{2}}(i - 1) \tag{140}$$

The unitary conditions for the matrix are easily verified.

## REFERENCES


Akhiezer, N. I., and Glazman, I. M. (1963). *Theory of Linear Operators in Hilbert Space, Vol. 2*, Ungar, New York.

Blank, J., Exner, P., and Havlicek, M. (1994). *Hilbert Space Operators in Quantum Physics*, American Institute of Physics, New York.

Busch, P., Grabowski, M., and Lahti, P. (1995). *Operational Quantum Physics*, Springer-Verlag, Berlin.

Devoret, M. H., Esteve, D., and Urbina, U. (1992). *Nature*, **360**, 547.

Exner, P., and Seba, P. (1987). *Journal of Mathemtical Physics*, **28**, 386.

Exner, P., and Seba, P. (1989a). *Reports on Mathematical Physics*, **28**, 7.

Exner, P., and Seba, P. (1989b). Quantum junctions and the self-adjoint extensions theory, in *Applications of Self-Adjoint Extensions in Quantum Physics*, P. Exner and P. Seba, eds., Springer-Verlag, Berlin.

Exner, P., Seba, P., and Stovicek, P. (1989). Quantum waveguides, in *Applications of Self-Adjoint Extensions in Quantum Physics*, P. Exner and P. Seba, eds., Springer-Verlag, Berlin.

Fano, G. (1971). *Mathematical Methods of Quantum Mechanics*, McGraw-Hill, New York.

Feynman, R. P. (1965). *The Feynman Lectures of Physics*, *Vol. III*, Addison-Wesley, Reading, Massachusetts, Section 21-9.

Feynman, R. P. (1972). *Statistical Mechanics*, Benjamin, Reading, Massachusetts, p. 304.

Gough, C. E. (1991). In *High Temperature Superconductivity: Proceedings of the 39th Scottish Universities Summer School in Physics, St Andrews, June 1991*, D. P. Tunstall and W. Barford, eds., Hilger, Bristol, England, p. 50.

Harrison, F. E. and Wan, K. K. (1997). *Journal of Physics A* **30**, 4731–4755.

Hudson, V., and Pym, S. J. (1980). *Applications of Functional Analysis and Operator Theory*, Academic Press, London, p. 269.

Josephson, B. D. (1962). *Physics Letters* **1**, 251.

Mandl, F. (1992). *Quantum Mechanics*, Wiley, New York, Section 2.2.

Reed, M., and Simon, B. (1975). *Fourier Analysis, Selfadjointness*, Academic Press, New York, pp. 144–145.

Richtmyer, R. D. (1978). *Principles of Advanced Mathematical Physics*, Vol. **1**, Springer-Verlag, New York.
Rose-Innes, A. C., and Rhoderick, E. H. (1969). *Introduction to Superconductivity*, Pergamon Press, Oxford.
Schroeck, E. F., Jr. (1996). *Quantum Mechanics on Phase Space*, Kluwer, Dordrecht.
Selleri, F., and Tarozzi, F. (1981). *Nuovo Cimento*, **4**(2).
Tilley, D. R., and Tilley, J. (1990). *Superfluidity and Superconductivity*, Hilger, Bristol, England, pp. 38–39.
Wan, K. K., and Fountain, R. H. (1996). *Foundations of Physics*, **26**, 1165.
Wan, K. K., and Harrison, F. E. (1993). *Physics Letters A*, **174**, 1.
Wan, K. K., and McLean, D. R. (1984). *Journal of Physics A* **17**, 835, 2363.
Wan K. K., Fountain, R. H., and Tao, Z. Y. (1995). *Journal of Physics A* **28**, 2379.
Weidman, J. (1980). *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York.
Wheeler, J. A., and Zurek, W. H., eds. (1983). *Quantum Theory and Measurement*, Princeton University Pres., Princeton, New Jersey.
Wollman, D. A., *et al.*, (1993). *Physics Review Letters*, **71**, 2134.